International Journal of Theoretical Physics, Vol. 42, No. 4, April 2003 (© 2003)

Kaluza–Klein Black-Hole Entropy by Quantum Statistics

Zhao Ren,[1,2] Wu Yueqin,[1] and Zhang Lichun[1]

Received December 11, 2002

By using the method of quantum statistics, we directly derive the partition functions of bosonic and fermionic field in Kaluza–Klein black hole with axial symmetry. Then via the improved brick-wall method, membrane model, we obtain that the entropy of bosonic and fermionic field in black hole is proportional to the area of horizon. In our result, the stripped term and the divergent logarithmic term no longer exist. The problem that the state density is divergent around the horizon doesn't exist either. We also give the influence of the spining degeneracy of particles on the entropy of black hole. We offer a new, simple, and direct way of calculating the entropy of different complicated black holes.

KEY WORDS: quantum statistics; brick-wall method; membrane model; entropy of black hole.

1. INTRODUCTION

Entropy of a black hole is one of the important subjects in theoretical physics. Since entropy has statistical meaning, the understanding of entropy involves the sense of the microscopic essence of the black hole. Fully understanding of it needs a good quantum gravitation theory. However, at present the work of it is not satisfying. The statistical origin of the black hole is not solved yet (Liberati, 1997). On the other hand, many literatures gave the same result that the entropy of the black hole is proportional to the area of horizon (Cai *et al.*, 1999; Cognola and Lecca, 1989; Frolov *et al.*, 1996; Hochberg *et al.*, 1993; Jing and Yan, 2001; Padmanaban, 1999; Solodukhin, 1995; 't Hooft, 1985). The most frequently used method among them is the brick-wall method advanced by 't Hooft (1985). This method is used to study the statistical properties of scalar field and Dirac field in various black holes (Lee and Kim, 1996; Liu and Zhao, 2000; Shen and Chen, 2000) and it is found that the general expression of the black hole's entropy consists

[1] Department of Physics, Yanbei Normal Institute, Datong, People's Republic of China.
[2] To whom correspondence should be addressed at Department of Physics, Yanbei Normal Institute, 037000 Datong, People's Republic of China.

0020-7748/03/0400-0809/0

of a term that is proportional to the area of its event horizon and a divergent logarithmic term that is not proportional to the area of event horizon. However, it is doubted that, first, the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole; second, the state density near the event horizon is divergent; third, the logarithmic term is left out and L^3 is considered as the contribution of distant vacuum surrounding the system; fourth, the wave function of scalar or Dirac field is solved approximately. The above mentioned problems with the original brick-wall method are unnatural and insurmountable.

It is known that the entropy of the black hole is proportional to the area of horizon and the existence of the horizon is the basic property of the black hole. It is proved that the existence of the horizon generally results in Hawking effect (Zhao, 1981). And whetehr there is the entropy of the black hole or not relates to the existence of the horizon (Gibbons and Hawking, 1977). Then it reveals that it is natural that the entropy of the black hole is proportional to the area of horizon. Its value has nothing to do with the radiation field outside the horizon. And the horizon only has the property of the two-dimensional membranne in three-dimensional space. Does the number of quantum states of the two-dimensional membrane correspond to the entropy of the black hole? If it does, calculating the entropy of the membrane will be the key issue.

We derive the bosonic and fermionic partition functions in Kaluza–Klein black holes directly by quantum statistical method (Zhao *et al.*, 2001) and obtain the integral expression of the system's entropy. Then we use the membrane model (Wu *et al.*, 2001; Zhao *et al.*, 2001a,b, 2002) to calculate entropy. As a result, the term left out in original brick-wall method no longer exists. The problem that the state density near the event horizon is divergent doesn't exist either. We also consider the spinning degeneracy of radiational particles. In the whole process, the physical idea is clear, calculation is simple, and the result is reasonable. It offers a neat way of studying black hole's entropy. In this paper, we take the simplest functional form of the temperature ($C = \hbar = K_B = 1$).

2. KALUZA–KLEIN BLACK HOLE

The line element in Kaluza–Klein space-time is given by (Frolov and Novikov, 1993)

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2 \sin^2\theta}{B\Sigma}\,dt^2 - 2a\sin^2\theta\,\frac{1}{\sqrt{1-\eta^2}}\,\frac{Z}{B}\,dt\,d\phi \\ &+ \left[B(r^2+a^2) + a^2\sin^2\theta\,\frac{Z}{B}\right]\sin^2\theta\,d\phi^2 \\ &+ \frac{B\Sigma}{\Delta}\,dr^2 + B\Sigma\,d\theta^2, \end{aligned} \tag{2.1}$$

where

$$\Delta = r^2 - 2\mu r + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad Z = \frac{2\mu r}{\Sigma}, \quad B = \left(1 + \frac{\eta^2 Z}{1-\eta^2}\right)^{1/2}.$$

The relations among mass, electric charge, and angular momentum and μ, η are respectively

$$M = \mu\left[1 + \frac{\eta^2}{2(1-\eta^2)}\right], \quad Q = \frac{\mu\eta}{1-\eta^2}, \quad J = \frac{\mu a}{\sqrt{1-\eta^2}},$$

where $r_\pm = \mu \pm \sqrt{\mu^2 - a^2}$ are the locations of outer and inner horizons, respectively.

$$A_\pm = 4\pi\frac{r_\pm^2 + a^2}{\sqrt{1-\eta^2}}. \tag{2.2}$$

The Hawking radiation temperature of the black hole is

$$T_+ = \frac{\sqrt{(1-\eta^2)(\mu^2 - a^2)}}{2\pi(r_\pm^2 + a^2)}. \tag{2.3}$$

3. THE BOSONIC ENTROPY

The natural radiation temperature (Lee and Kim, 1996; Tolman, 1934) got by the observer at rest at an infinite distance is

$$T = \frac{T_+}{\sqrt{-g'_{u}}}, \tag{3.1}$$

where T_+ is the equilibrium temperature and

$$g'_{tt} = \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} = -\frac{\Delta B}{(1-\eta^2)B^2(r^2+a^2) + a^2\sin^2\theta Z}. \tag{3.2}$$

For bosonic gas, we calculate the partition function as

$$\ln Z = -\sum_t g_i \ln(1 - e^{-\beta\varepsilon_t}). \tag{3.3}$$

In unit volume, the number of quantum states with the energy between ε and $\varepsilon + d\varepsilon$ or the frequency between v and $v + dv$ is as follows:

$$g(v)\,dv = j4\pi v^2\,dv, \tag{3.4}$$

where j is the spin degeneracy of particles. Since in the space-time (2.1), the area of two-dimensional curved surface at random point r is

$$A(r) = \int dA = \int \sqrt{g}\,d\theta\,d\varphi, \tag{3.5}$$

where

$$g = \begin{vmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = g_{\theta\theta} g_{\varphi\varphi}.$$

The volume of the lamella at random point r outside the horizon is as follows:

$$dV = A(r)\sqrt{g}_{rr}\, dr. \tag{3.6}$$

So, the partition function of the system at the lamella with random thickness at point r outside the horizon is as follows:

$$\begin{aligned} \ln Z &= \int A(r)\sqrt{g_{rr}}\, dr \sum_i g_i \sum_{n=1} \frac{1}{n} e^{-n\beta\varepsilon_i} \\ &= j4\pi \int A(r)\sqrt{g_{rr}}\, dr \sum_{n=1} \frac{1}{n} \int_0^\infty e^{\frac{nh\nu}{T}} \nu^2\, d\nu = j\frac{1}{90}\pi^2 \int \frac{A(r)\sqrt{g_{rr}}\, dr}{\beta^3} \\ &= j\frac{\pi^2}{90} \int \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}}\, dr\, d\theta\, d\varphi}{\beta^3}, \end{aligned} \tag{3.7}$$

where $1/\beta = T$, using the relation between entropy and partition function

$$S = \ln Z - \beta_0 \frac{\partial \ln Z}{\partial \beta_0}, \tag{3.8}$$

we have

$$\begin{aligned} S_b &= j\frac{2}{45}\pi^2 \frac{1}{\beta_0^3} \int \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}}\, dr}{(-g'_{tt})^{3/2}}\, d\theta\, d\varphi = j\frac{2\pi^2}{45\beta_0^3} \int d\theta\, d\varphi \\ &\times \int \frac{[(r^2+a^2)\left(r^2+a^2+\frac{2\mu r\eta^2}{l-\eta^2}\right) - (r-r_+)(r-_-)a^2\sin^2\theta]^2}{B(r-r_+)^2(r-r_-)^2(r^2+a^2\cos^2\theta)} \sin\theta\, dr \end{aligned} \tag{3.9}$$

where

$$\beta_0 = \frac{1}{T_+} = \frac{4\pi(r_+^2+a^2)}{(r_+-r_-)\sqrt{1-\eta^2}}, \quad \text{and} \quad \beta = \beta_0\sqrt{-g_{tt}}.$$

In the above integral, we take the r integral region $[r_+ + \varsigma, r_+ + N\varsigma]$, where ς is a small nonnegative quantity, N is a constant larger than 1 (Zhao *et al.*, 2001a). So we have,

$$\begin{aligned} S_b &= j\frac{2\pi^2}{45\beta_0^3} \int d\theta\, dd\varphi \\ &\int_{r_++\varsigma}^{r_++N_\varsigma} \frac{[(r^2+a^2)(r^2+a^2+\frac{2\mu r\eta^2}{l-\eta^2}) - (r-r_+)(r-r_-)a^2\sin^2\theta]^2}{B(r-r_+)^2(r-_-)^2(r^2+a^2\cos^2\theta)} \sin\theta\, dr \end{aligned}$$

$$= j\frac{4\pi^3}{45\beta_0^3}\int_0^\pi \frac{(r_+^2+a^2)^4}{(1-\eta^2)^2(r_+-r_-)^2(r_+^2+a^2\cos^2\theta)B(r_+)}$$
$$\left[\frac{N-1}{N\varsigma}\right]\sin\theta d\theta + G(r_+, N, \varsigma). \tag{3.10}$$

From (3.17) in the 't Hooft (1995), we know when $N\varsigma = L \gg r_+$, if we take

$$\varsigma = \frac{T_+}{90}, \tag{3.11}$$

we obtain that the entropy of black hole is proportional to the area of its horizon. In order to let the calculated entropy be independent of parameters N and ς introduced in (3.10), we take

$$\varsigma = \frac{T_+}{90}\frac{\alpha}{\sin\alpha\ \cos\alpha}\frac{1}{B(r_+)(1-\eta^2)^{1/2}}\frac{N-1}{N}. \tag{3.12}$$

Then (3.10) can be reduced to

$$S_b = j\frac{\sin\alpha\cos\alpha}{\alpha}\int_0^\pi \frac{\pi(r^2+a^2)^2}{2(1-\eta^2)^{1/2}(r_+^2+a^2\cos^2\theta)}\sin\theta d\theta + G(r_+, N, \varsigma)$$
$$= \frac{\pi(r_+^2+a^2)}{(1-\eta^2)^{1/2}} + G(r_+, N, \varsigma) = j\frac{1}{4}A_+ + G(r_+, N, \varsigma). \tag{3.13}$$

where

$$G(r_+, N, \varsigma) = j\frac{4\pi^3}{45\beta_0^3}\int_0^\pi \left[f_1(r_+) - \frac{2(r_+^2+a^2)^2a^2\sin^2\theta}{(1-\eta^2)(r_+-r_-)(r_+^2+a^2\cos^2\theta)B(r_+)}\right]$$
$$\times \sin\theta\, d\theta \ln N + j\frac{4\pi^3}{45\beta_0^3}\int_0^\pi \sin\theta d\theta \int_{r_++\varsigma}^{r_++N\varsigma} F(r)\, dr, \tag{3.14}$$

$$F(r) = \sum_{n=2}^{\infty}\frac{f_1^{(n)}(r_+)}{n!}(r-r_+)^{n-2} - \sum_{n=1}^{\infty}\frac{f_1^{(n)}(r_+)}{n!}(r-r_+)^{n-1} + f_3(r), \tag{3.15}$$

and

$$f_1(r) = \frac{(r^2+a^2)^2\left(r^2+a^2+\frac{2\mu r\eta^2}{1-\eta^2}\right)^2}{(r-r_-)^2(r^2+a^2\cos^2\theta)B(r)},$$
$$f_2(r) = \frac{2(r^2+a^2)\left(r^2+a^2+\frac{2\mu r\eta^2}{1-\eta^2}\right)^2 a^2\sin^2\theta}{(r-r_-)(r^2+a^2\cos^2\theta)B(r)},$$

$$f_3(r) = \frac{a^4 \sin^4 \vartheta}{B(r)(r^2 + a^2 \cos^2 \theta)}, \quad f_1'(r_+) = \left. \frac{df_1}{dr} \right|_{r=r_+},$$

$$f_2'(r_+) = \left. \frac{df_2}{dr} \right|_{r=r_+}, \quad \alpha = \text{arctg}\frac{a}{r_+}.$$

As $N \to 1$, $\varsigma \to 0$, and $N\varsigma \to 0$, that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole, and $F(r)$ is analytic function at $r = r_+$, we have $\lim_{N\to 1} G(r_+, N, \varsigma) \to 0$. The black hole's entropy is as follows:

$$S_b = j\frac{1}{4}A+, \tag{3.16}$$

where

$$A_+ = \frac{4\pi(r_+^2 + a^2)}{(1 - \eta^2)^{1/2}}$$

is the area of horizon. Since we let the integral upper limit and lower limit tend to the outer horizon, the entropy obtained in (3.16) is independent of the radiation field outside horizon. It only has the property of two-dimensional membrane in three-dimensional space. So the obtained entropy has the property of one-dimensional membrane. The existence of horizon is the basic property of black hole. It has already been proved that the general existence of horizon leads to the Hawking effect (Zhao, 1981). And whether there is black hole's entropy or not directly involves the existence of horizon (Gibbons and Hawking, 1977). So the entropy in (3.16) should be black hole's entropy. When $j = 1$ for radiation particles, we obtain that black hole's entropy is a quarter of the area of horizon. When $j \neq 1$, we can take j into consideration in the parameters N and ς in (3.12) to make sure that black hole's entropy is a quarter of the area of horizon.

4. FERMIONIC ENTROPY

For Fermionic gas, the grand partition function is as follows:

$$\ln Z = \sum_l g_1 \ln(1 + e^{-\beta\varepsilon_l}). \tag{4.1}$$

From (3.7), we obtain

$$\begin{aligned} \ln Z &= \int A(r)\sqrt{g_{rr}}\, dr \sum_i g_i \sum_{n=1} \frac{(-1)^{n-1}}{n} e^{-n\beta\varepsilon_i} \\ &= \omega 4\pi \int A(r)\sqrt{g_{rr}}\, dr \sum_{n=1} \frac{(-1)^{n-1}}{n} \int_0^\infty e^{\frac{nh\nu}{T}} \nu^2 d\nu \end{aligned}$$

$$= \omega \frac{\pi^2}{90} \frac{7}{8} \int \frac{A(r)\sqrt{g_{rr}}\, dr}{\beta^3} = \omega \frac{\pi^2}{90} \frac{7}{8} \int \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}} dr\, d\theta\, d\varphi}{\beta^3}, \quad (4.2)$$

where ω is the spinning degeneracy of fermionic particles. Using the result of part three, we can get the fermionic entropy as follows:

$$S_f = \omega \frac{7}{8} \frac{1}{4} A_+. \quad (4.3)$$

5. CONCLUSION

In the above analysis, we derive partition functions of various fields in black hole with different temperature on horizon surface directly by using the statistical method. We avoid the difficulty in solving wave equation. Since we use the improved brick-wall method, membrane model, to calculate the entropy of various fields, the problem that the state density is divergent around horizon does not exist any more. In our calculation, as $N \to 1$, $\varsigma \to 0$ and $N\varsigma \to 0$, that is, the inner and outer "brick walls" both approach the outer horizon of black hole. From (3.16) and (4.3), we know that the divergent logarithmic term and L^3 term in the original brick-wall method no longer exist. The obtained entropy is proportional to the area of its horizon.

In above analysis, we know that by using the statistical and membrane model methods, the doubt that why the entropy of the scalar or Dirac field outside the event horizon is the entropy of black hole in the original brick-wall method doesn't exist and the complicated approximations in solution is avoided. In the whole process, the physical idea is clear; the calculation is simple; and the result is reasonable. We also consider the influence of the spinning degeneracy of particles on the entropy. In calculation of entropy of Kaluza–Klein black hole, the relation between N and ς is related to angle θ. When the space-time reduces to Reissner–Nordstrom space-time, the relation between N and ς also can be reduced to the result in the Zhao *et al.* (2001a).

ACKNOWLEDGMENT

This project was supported by the Shanxi Natural Science Foundation of China.

REFERENCES

Cai, R. G., Ji, J. Y., and Soh, K. S. (1998). Action and entropy f black hole in spacetime with a cosmological constant. *Classical and Quantum Gravity* **15**, 2783.

Cognola, G. and Lecca, P. (1998). Electromagnetic field in Schwarzschild and Reissner–Nordstrom geometry. *Physical Review D* **57**, 1108.

Frolov, V. P., Fursaev, D. V., and Zelnikov, A. I. (1996). Black hole entropy: Off shell versus on shell. *Physical Review D* **54**, 2711.

Frolov, V. and Novikov, I. (1993). Dynamical origin of entropy of a black hole. *Physical Review D* **48**, 4545.

Gibbons, G. W. and Hawking, S. W. (1977). Action integrals and partition function in quantum gravity. *Physical Review D* **15**, 2752.

Hochberg, D.,Kephart, T. W., and York, T. W. (1993). Positivity of entropy in the semiclassical theory of black hole and radiation. *Physical Review D* **48**, 479.

Jing, J. L. and Yan, M. L. (2001). Quantum entropy of the Kerr black hole arising from gravitational perturbation. *Physical Review D* **64**, 064015.

Lee, M. H. and Kim, J. K. (1996). Entropy of a quantum field in rotating black hole. *Physical Review D* **54**, 3904.

Liberati, S. (1997). Problems in black-hole entropy interpretation. *Nuovo Cimento B* **112**, 405.

Liu, W. B. and Zhao, Z. (2000). Entropy of Dirac field in a Kerr–Newman black hole. *Physical Review D* **61**, 063003.

Padmanaban, T. (1989). Phase volume occupied by a test particlearound an incipient black hole. *Physics Letters A.* **136**, 203.

Shen, G. Y. and Chen, D. M. (2000). The quantum entropy in Horowitz–Strominger black hole background. *General Relativity and Gravitation* **32**, 2269.

Solodukhin, S. N. (1995). Conical singularity and quantum correction to the entropy of a black hole. *Physical Review D* **51**, 609.

't Hooft, G. (1985). On the quantum structure of a black hole. *Nuclear Physics, B* **256**, 727.

Tolman, R. C. (1934). *Relativity, Thermodynamics and Cosmology*, Oxford University Press, Oxford.

Wu, Y. Q., Zhang, L. C., and Zhao, R. (2001). Black hole and cosmic entropy for Schwarzschild–de Sitter space-time. *International Journal of Theoretical Physics* **40**, 1001.

Zhao, Z. (1981). Hawking radiation in four dimensional static riemann spacetime. *Acta Physica Sinica* **30**, 1508. (in Chinese)

Zhao, R., Zhang, J. F., and Zhang, L. C. (2001a). Statistical entropy of Reissner–Nordstrom black hole. *Nuclear Physics B* **609**, 247.

Zhao, R., Zhang, J. F., and Zhang, L. C. (2001b). Entropy of Schwarzschild–de Sitter black hole in non-thermal-equilibrium. *Modern Physics Letters A* **16**, 719.

Zhao, R., Zhang, J. F., and Zhang, L. C. (2002). Entropy of dilatonic black hole. *International Journal of Theoretical Physics*, **41**, 1369.